1. Eq. (9.38):

$$H = \begin{pmatrix} \frac{1}{2}\hbar\omega_0 & e\mathcal{E}_0 \vec{r}_{eg}^{\,*}\cdot(\vec{\epsilon}\, e^{i\omega t} + \vec{\epsilon}^{\,*} e^{-i\omega t}) \\ \underbrace{e\mathcal{E}_0 \vec{r}_{eg}\cdot(\vec{\epsilon}\, e^{i\omega t} + \vec{\epsilon}^{\,*} e^{-i\omega t})}_{A} & \underbrace{-\frac{1}{2}\hbar\omega_0}_{-B} \end{pmatrix}$$

a) $|\psi'\rangle = U|\psi\rangle$

$$\frac{d}{dt}|\psi'\rangle = \left(\frac{dU}{dt}\right)|\psi\rangle + U\frac{d}{dt}|\psi\rangle$$

$$= \frac{dU}{dt}|\psi\rangle + \frac{1}{i\hbar}UH|\psi\rangle$$

$$i\hbar\frac{d}{dt}|\psi'\rangle = i\hbar\frac{dU}{dt}U^\dagger U|\psi\rangle + UHU^\dagger U|\psi\rangle$$

$$i\hbar\frac{d}{dt}|\psi'\rangle = i\hbar\left(\frac{dU}{dt}\right)U^\dagger|\psi'\rangle + UHU^\dagger|\psi'\rangle = H'|\psi'\rangle$$

$$H' = UHU^\dagger + i\hbar\left(\frac{dU}{dt}\right)U^\dagger$$

b) $H = \begin{pmatrix} B & A^* \\ A & -B \end{pmatrix}$ $\qquad U = \begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix}$

$$\frac{dU}{dt} = \frac{i\omega}{2}\begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & -e^{-i\omega t/2} \end{pmatrix}$$

$$H' = \begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix}\begin{pmatrix} B & A^* \\ A & -B \end{pmatrix}\begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{pmatrix} - \frac{\hbar\omega}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix}\begin{pmatrix} Be^{-i\omega t/2} & A^* e^{i\omega t/2} \\ Ae^{-i\omega t/2} & -Be^{i\omega t/2} \end{pmatrix} - \frac{\hbar\omega}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$H' = \begin{pmatrix} B & A^* e^{i\omega t} \\ A e^{-i\omega t} & -B \end{pmatrix} - \frac{\hbar\omega}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}\hbar\omega_0 - \frac{1}{2}\hbar\omega & e\mathcal{E}_0 \vec{r}_{eg}^{\,*}\cdot\vec{\epsilon}\, e^{2i\omega t} + e\mathcal{E}_0 \vec{r}_{eg}^{\,*}\cdot\vec{\epsilon}^{\,*} \\ e\mathcal{E}_0 \vec{r}_{eg}\cdot\vec{\epsilon} + e\mathcal{E}_0 \vec{r}_{g}\cdot\vec{\epsilon}^{\,*} e^{-2i\omega t} & -\frac{1}{2}\hbar\omega_0 + \frac{1}{2}\hbar\omega \end{pmatrix}$$

The approximation is now that $e^{2i\omega t}$ and $e^{-2i\omega t}$ are <u>fast</u> oscillating terms that average <u>zero</u> over long enough time scales. So the new Hamiltonian becomes

$$H' = \begin{pmatrix} \frac{1}{2}\hbar(\omega_0-\omega) & e\mathcal{E}_0 \vec{r}_{eg}^{\,*}\cdot\vec{\epsilon}^{\,*} \\ e\mathcal{E}_0 \vec{r}_{eg}\cdot\vec{\epsilon} & -\frac{1}{2}\hbar(\omega_0-\omega) \end{pmatrix} \equiv \frac{\hbar}{2}\begin{pmatrix} \nu & \Omega^* \\ \Omega & -\nu \end{pmatrix}$$

with $\nu = \omega_0 - \omega$ and $\Omega = \dfrac{2e\mathcal{E}_0}{\hbar}\vec{r}_{eg}\cdot\vec{\epsilon}$.

This gives equation (9.39), as required.

2 a) $H_{JC} = \frac{1}{2}\hbar\omega_0\sigma_3 + \hbar\omega a^\dagger a + g\sigma_+ a + g^*\sigma_- a^\dagger$

For every photon number state n, there are two atomic states $|g\rangle$ and $|e\rangle$. The Hamiltonian does not mix different photon number states while keeping the atomic state unchanged.

The state $|g,n\rangle$ will be taken to itself by the first two terms in H_{JC} (up to a factor), and the third term takes the state to $|e, n-1\rangle$:

$$g\sigma_+ a|g,n\rangle = g\sqrt{n}\,|e, n-1\rangle$$

The last term kills $|g,n\rangle$ since $\sigma_-|g\rangle = 0$.

Similarly, $|e, n-1\rangle$ will be taken to itself or $|g,n\rangle$:

$$g^*\sigma_- a^\dagger |e, n-1\rangle = g^*\sqrt{n}\,|g,n\rangle$$

On the basis $\{|e, n-1\rangle, |g,n\rangle\}$ the Hamiltonian becomes:

$$\langle e, n-1|H_{JC}|e, n-1\rangle = \langle e, n-1|\left(\tfrac{1}{2}\hbar\omega_0 + \hbar\omega(n-1)\right)|e, n-1\rangle$$

$$= \hbar\left(\frac{\omega_0}{2} + (n-1)\omega\right)$$

$$\langle g,n|H_{JC}|g,n\rangle = \langle g,n|\left(-\tfrac{1}{2}\hbar\omega_0 + \hbar\omega n\right)|g,n\rangle = \hbar\left(-\frac{\omega_0}{2} + n\omega\right)$$

This leads to

$$H_n = \begin{pmatrix} \frac{1}{2}\hbar\omega_0 + (n-1)\hbar\omega & g^*\sqrt{n} \\ g\sqrt{n} & -\frac{1}{2}\hbar\omega_0 + n\hbar\omega \end{pmatrix}$$

b) $\det\left(H_n - \lambda \mathbb{I}\right) = (a-\lambda)(b-\lambda) - n|g|^2 = 0$

$a = \frac{1}{2}\hbar\omega_0 + (n-1)\hbar\omega$ and $b = -\frac{1}{2}\hbar\omega_0 + n\hbar\omega$

$$ab - (a+b)\lambda + \lambda^2 - n|g|^2 = 0$$

$$ab = \left(\frac{1}{2}\hbar\omega_0 + (n-1)\hbar\omega\right)\left(-\frac{1}{2}\hbar\omega_0 + n\hbar\omega\right)$$

$$= -\frac{1}{4}\hbar^2\omega_0^2 + \frac{n}{2}\hbar^2\omega_0\omega - \frac{n-1}{2}\hbar^2\omega_0\omega + n(n-1)\hbar^2\omega^2$$

$$= \hbar^2\left(n(n-1)\omega^2 + \frac{1}{2}\omega\omega_0 - \frac{1}{4}\omega_0^2\right)$$

$$a+b = (2n-1)\hbar\omega$$

Therefore:

$$\lambda^2 - (2n-1)\hbar\omega\,\lambda + \hbar^2\left(n(n-1)\omega^2 + \frac{1}{2}\omega\omega_0 - \frac{1}{4}\omega_0^2\right) - n|g|^2 = 0$$

$$\lambda_{1,2} = \frac{(2n-1)\hbar\omega \pm \sqrt{(2n-1)^2\hbar^2\omega^2 - 4\hbar^2\left(n(n-1)\omega^2 + \frac{1}{2}\omega\omega_0 - \frac{1}{4}\omega_0^2\right) + 4n|g|^2}}{2}$$

$$= \left(n-\frac{1}{2}\right)\hbar\omega \pm \hbar\omega\sqrt{\left(n-\frac{1}{2}\right)^2 - n(n-1) - \frac{1}{2}\frac{\omega_0}{\omega} + \frac{1}{4}\left(\frac{\omega_0}{\omega}\right)^2 + \frac{n|g|^2}{\hbar^2\omega^2}}$$

$$= \left(n-\frac{1}{2}\right)\hbar\omega \pm \hbar\omega\sqrt{\frac{1}{4} - \frac{1}{2}\frac{\omega_0}{\omega} + \frac{1}{4}\frac{\omega_0^2}{\omega^2} + \frac{n|g|^2}{\hbar^2\omega^2}}$$

$$= \left(n-\frac{1}{2}\right)\hbar\omega \pm \hbar\omega\sqrt{\frac{1}{4}\left(1-\frac{\omega_0}{\omega}\right)^2 + \frac{n|g|^2}{\hbar^2\omega^2}}$$

check: $|g|^2 \to 0$

$$\lambda_{1,2} = \left(n-\frac{1}{2}\right)\hbar\omega \pm \frac{1}{2}\hbar\omega\left(1-\frac{\omega_0}{\omega}\right) = \left(n-\frac{1}{2}\right)\hbar\omega \pm \frac{1}{2}\hbar\omega \mp \frac{1}{2}\hbar\omega_0$$

$$\lambda_1 = n\hbar\omega - \frac{1}{2}\hbar\omega_0$$

$$\lambda_2 = (n-1)\hbar\omega + \frac{1}{2}\hbar\omega_0$$

which are the energies of $|g,n\rangle$ and $|e,n-1\rangle$ resp.

eigenstates

$$H_n \vec{v}_1 = \lambda_1 \vec{v}_1 \quad \text{and} \quad H_n \vec{v}_2 = \lambda_2 \vec{v}_2$$

Solve these to get

$$\vec{v}_1 = \begin{pmatrix} -\hbar\omega + \hbar\omega_0 + \sqrt{\hbar^2\omega^2 - 2\hbar^2\omega\omega_0 + \hbar^2\omega_0^2 + 4n|g|^2} \\ 2g\sqrt{n} \end{pmatrix}$$

$$\vec{v}_2 = \begin{pmatrix} -\hbar\omega + \hbar\omega_0 - \sqrt{\hbar^2\omega^2 - 2\hbar^2\omega\omega_0 + \hbar^2\omega_0^2 + 4n|g|^2} \\ 2g\sqrt{n} \end{pmatrix}$$

These are not normalised.

c) We know that the state at some time $t > 0$ is a superposition of $|e, n-1\rangle$ and $|g, n\rangle$:

$$|\psi(t)\rangle = \alpha\,|e, n-1\rangle + \beta\,|g, n\rangle$$

Since $i\hbar \frac{d}{dt}|\psi(t)\rangle = H_n |\psi(t)\rangle$, we have

$$i\hbar\dot{\alpha}\,|e, n-1\rangle + i\hbar\dot{\beta}\,|g, n\rangle = \alpha H_n |e, n-1\rangle + \beta H_n |g, n\rangle$$

$$H_n|e, n-1\rangle = \left(\tfrac{1}{2}\hbar\omega_0 + (n-1)\hbar\omega\right)|e, n-1\rangle + g^*\sqrt{n}\,|g, n\rangle$$

$$H_n|g, n\rangle = \left(-\tfrac{1}{2}\hbar\omega_0 + n\hbar\omega\right)|g, n\rangle + g\sqrt{n}\,|e, n-1\rangle$$

We can write this as

$$i\hbar\left(\dot{\alpha}\,|e, n-1\rangle + \dot{\beta}\,|g, n\rangle\right) = \alpha\left(\tfrac{1}{2}\hbar\omega_0 + (n-1)\hbar\omega\right)|e, n-1\rangle + \alpha g^*\sqrt{n}\,|g, n\rangle + \beta\left(-\tfrac{1}{2}\hbar\omega_0 + n\hbar\omega\right)|g, n\rangle + \beta g\sqrt{n}\,|e, n-1\rangle$$

Since $|e,n-1\rangle$ and $|g,n\rangle$ are linearly independent the two differential equations must separately be satisfied:

$$i\hbar\dot{\alpha} = \alpha\left(\tfrac{1}{2}\hbar\omega_0 + (n-1)\hbar\omega\right) + \beta g\sqrt{n}$$

$$i\hbar\dot{\beta} = \beta\left(-\tfrac{1}{2}\hbar\omega_0 + n\hbar\omega\right) + \alpha g^*\sqrt{n}$$

These are two linear coupled differential equations:

$$i\hbar\frac{d}{dt}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\tfrac{1}{2}\hbar\omega_0 + (n-1)\hbar\omega & g^*\sqrt{n}\\ g\sqrt{n} & -\tfrac{1}{2}\hbar\omega_0 + n\hbar\omega\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = H_n\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

To solve this we diagonalise H_n. But we already did this in part b) and c):

$$i\hbar\frac{d}{dt}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}\lambda_1 v_1\\ \lambda_2 v_2\end{pmatrix} \quad \text{or} \quad \begin{aligned} v_1(t) &= e^{-i\lambda_1 t/\hbar}\, v_1(0)\\ v_2(t) &= e^{-i\lambda_2 t/\hbar}\, v_2(0)\end{aligned}$$

with
$$v_1 = \left(-\hbar\omega + \hbar\omega_0 + \sqrt{\hbar^2\omega^2 - 2\hbar^2\omega\omega_0 + \hbar^2\omega_0^2 + 4n|g|^2}\right)\alpha + 2g\sqrt{n}\,\beta$$

$$v_2 = \left(-\hbar\omega + \hbar\omega_0 - \sqrt{\hbar^2\omega^2 - 2\hbar^2\omega\omega_0 + \hbar^2\omega_0^2 + 4n|g|^2}\right)\alpha + 2g\sqrt{n}\,\beta$$

therefore $\alpha = \dfrac{v_1 - v_2}{2x}$ with $x = \sqrt{\hbar^2\omega^2 - 2\hbar^2\omega\omega_0 + \hbar^2\omega_0^2 + 4n|g|^2}$.

$$\alpha(t) = \frac{v_1(0)e^{-i\lambda_1 t/\hbar} - v_2(0)e^{-i\lambda_2 t/\hbar}}{2x}.$$

$$v_1(0) = -\hbar\omega + \hbar\omega_0 + x$$
$$v_2(0) = -\hbar\omega + \hbar\omega_0 - x$$

$$\alpha(t) = \frac{\hbar(\omega_0 - \omega)}{x}\,\frac{e^{-i\lambda_1 t/\hbar} - e^{-i\lambda_2 t/\hbar}}{2} + \frac{e^{-i\lambda_1 t/\hbar} + e^{-i\lambda_2 t/\hbar}}{2}.$$

Suppose $\omega = \omega_0$. Then $\alpha(t) = \dfrac{e^{-i\lambda_1 t/\hbar} + e^{-i\lambda_2 t/\hbar}}{2}$

We can write this as

$$\alpha(t) = e^{-i(\lambda_1+\lambda_2)t/\hbar} \cos\left[\frac{(\lambda_1-\lambda_2)t}{\hbar}\right]$$

Up to some phase ϕ $\beta(t)$ will be $\sin\left[\dfrac{(\lambda_1-\lambda_2)t}{\hbar}\right]$:

$$|\psi(t)\rangle = \cos\left[\frac{(\lambda_1-\lambda_2)t}{\hbar}\right]|e, n-1\rangle + e^{i\phi}\sin\left[\frac{(\lambda_1-\lambda_2)t}{\hbar}\right]|g, n\rangle$$

d) Relative entropy: trace out the cavity field and calculate the entropy of the remaining state of the atom:

$$\rho = \cos^2\Big[\underbrace{\frac{(\lambda_1-\lambda_2)t}{\hbar}}_{\theta}\Big]|e,n-1\rangle\langle e,n-1| + \sin^2\left[\frac{(\lambda_1-\lambda_2)t}{\hbar}\right]|g,n\rangle\langle g,n|$$

$$S(\rho) = -\cos^2\theta \log\cos^2\theta - \sin^2\theta\log\sin^2\theta$$

e) Maximal entanglement : $S(\rho)$ is maximal $\rightarrow \cos^2\theta = \frac{1}{2}$

$$\theta = \frac{\pi}{4} = \frac{(\lambda_1-\lambda_2)t}{\hbar} \rightarrow t = \frac{\hbar\pi}{4(\lambda_1-\lambda_2)}.$$

For $\omega = \omega_0$: $\lambda_1 - \lambda_2 = 2\sqrt{n}\,|g|$.

So the stronger the interaction, the faster we create maximal entanglement.

When $\omega \neq \omega_0$ the problem is considerably more complicated.