1) $$\psi_A(r_1, r_2, r_3) = \frac{1}{\sqrt{6}} \begin{vmatrix} \psi_1(r_1) & \psi_1(r_2) & \psi_1(r_3) \\ \psi_2(r_1) & \psi_2(r_2) & \psi_2(r_3) \\ \psi_3(r_1) & \psi_3(r_2) & \psi_3(r_3) \end{vmatrix}$$

$$= \frac{1}{\sqrt{6}} \Big[\psi_1(r_1)\big(\psi_2(r_2)\psi_3(r_3) - \psi_2(r_3)\psi_3(r_2)\big) - \psi_2(r_1)\big(\psi_1(r_2)\psi_3(r_3) - \psi_1(r_3)\psi_3(r_2)\big) + \psi_3(r_1)\big(\psi_1(r_2)\psi_2(r_3) - \psi_1(r_3)\psi_2(r_2)\big) \Big]$$

Every triple $\psi_1 \psi_2 \psi_3$ has differing indices, which means that no electrons can be in the state $\psi_1(r_1)\,\psi_1(r_2)\,\psi_2(r_3)$.

same state

2a) $$p_j(n) = \mathrm{Tr}\big(|n_j\rangle\langle n_j|\rho\big)$$

$$\rho = \frac{\sum_{n'_j=0}^{\infty} e^{\beta(\mu-\epsilon_j)n'_j} |n'_j\rangle\langle n'_j|}{-e^{+\beta(\mu-\epsilon_j)} + 1}$$

n'_j primed to resolve notation clash

$$p_j(n) = \mathrm{Tr}\big(|n_j\rangle\langle n_j|\rho\big) = \frac{e^{\beta(\mu-\epsilon_j)n_j}}{1 - e^{\beta(\mu-\epsilon_j)n_j}}$$

$$= \frac{1}{-1 + e^{-\beta(\mu-\epsilon_j)n_j}} .$$ for bosons.

different from before (8.49)

2b) Fermions : $p_j(n) = \dfrac{1}{e^{-\beta(\mu-E_j)n_j}+1}$

different from (8.52)

3a) $\langle n_0 \rangle = \dfrac{1}{e^{-\beta(\mu-E_0)n_0}-1}$

$\mu \to E_0$ means that $\langle n_0 \rangle \to \infty$

As μ passes through E_0 we have a phase transition

b) $\langle n_{thermal} \rangle = \sum_{j=1}^{\infty} \langle n_j \rangle$,

E_0 is the lowest energy, so while $\mu \to E_0$ makes $\langle n_0 \rangle$ infinite , $\langle n_j \rangle$ remains finite $(j>0)$

therefore $\lim_{\mu \to E_0} \dfrac{\langle n_0 \rangle}{\langle n_{thermal} \rangle} \to 1.$

All the particles go into the ground state.

c) Bose-Einstein condensation.

4a) $K_+ = a_1^\dagger a_2^\dagger$ and $K_- = a_1 a_2$

$$[K_-, K_+] = [a_1 a_2, a_1^\dagger a_2^\dagger] = a_1[a_2, a_1^\dagger a_2^\dagger] + [a_1, a_1^\dagger a_2^\dagger]a_2$$

$$= a_1 a_1^\dagger + a_2^\dagger a_2 = a_1^\dagger a_1 + a_2^\dagger a_2 + 1 = 2K_0$$

$$[K_0, K_+] = \tfrac{1}{2}[a_1^\dagger a_1 + a_2^\dagger a_2 + 1, a_1^\dagger a_2^\dagger]$$

$$= \tfrac{1}{2}[a_1^\dagger a_1, a_1^\dagger a_2^\dagger] + \tfrac{1}{2}[a_2^\dagger a_2, a_1^\dagger a_2^\dagger]$$

$$= \tfrac{1}{2} a_1^\dagger a_2^\dagger + \tfrac{1}{2} a_2^\dagger a_1^\dagger = a_1^\dagger a_2^\dagger = K_+$$

$$[K_0, K_-] = \tfrac{1}{2}[a_1^\dagger a_1 + a_2^\dagger a_2 + 1, a_1 a_2]$$

$$= \tfrac{1}{2}[a_1^\dagger a_1, a_1 a_2] + \tfrac{1}{2}[a_2^\dagger a_2, a_1 a_2]$$

$$= -\tfrac{1}{2} a_1 a_2 - \tfrac{1}{2} a_2 a_1 = -a_1 a_2 = -K_- \qquad \square$$

b) $e^{\frac{r}{|r|}\tanh|r|\, a_1^\dagger a_2^\dagger}\, e^{-\ln(\cosh|r|)(a_1^\dagger a_1 + a_2^\dagger a_2 + 1)}\, e^{-\frac{r^*}{|r|}\tanh|r|\, a_1 a_2}|0\rangle$

$a_1|0\rangle = 0$ and $a_2|0\rangle = 0$ so $e^{\cdots a_1 a_2}|0\rangle = |0\rangle$

The middle exponential gives a ~~global phase~~ ~~and~~ normalistation.

$$e^{-\ln(\cosh|r|)} = \frac{1}{\cosh|r|} \qquad \frac{r}{|r|} = e^{i\phi}$$

~~global phase.~~

$$\text{So } |\psi\rangle = \frac{1}{\cosh|r|}\sum_{n=0}^{\infty} \frac{r}{|r|}\frac{\tanh^n|r|}{n!}(a_1^\dagger a_2^\dagger)^n|0\rangle$$

$$= \frac{1}{\cosh|r|}\sum_{n=0}^{\infty} e^{in\phi}\frac{\tanh^n|r|}{n!}\sqrt{n!}\cdot\sqrt{n!}\,|n,n\rangle$$

$$= \frac{1}{\cosh|r|}\sum_{n=0}^{\infty} e^{in\phi}\tanh^n|r|\,|n,n\rangle.$$

c) Trace out system 2:

$$\rho_1 = Tr_2\left(|\Psi\rangle\langle\Psi| \right)$$

$$= Tr_2\left(\sum_{nm} \frac{\tanh^{n+m}|r|}{\cosh^2|r|} e^{i(n-m)\phi} |n,n\rangle\langle m,m| \right)$$

$$= \sum_{n=0}^{\infty} \frac{\tanh^{2n}|r|}{\cosh|r|} |n\rangle\langle n|$$

This is diagonal, so the probabilities are just

$$p_1(n) = \frac{\tanh^{2n}|r|}{\cosh^2|r|} \equiv \frac{x^n}{y} \quad \left(\& \sum_n x^n = y \right)$$

$$S_1 = - \sum_n p_1(n) \ln p(n_1)$$

$$= - \sum_n \frac{x^n}{y} \ln \frac{x^n}{y} = - \sum_n \frac{x^n}{y}\left(\ln x^n - \ln y \right)$$

$$= - \sum_n \frac{n x^n}{y} \ln x + \sum_n x^n \frac{\ln y}{y}$$

$$= - \frac{x}{(1-x)^2} \frac{\ln x}{y} + \frac{1}{1-x} \frac{\ln y}{y}$$

$$- \frac{x}{(1-x)^2} = - \sinh^2|r| \quad \text{and} \quad \frac{1}{y(1-x)} = 1$$

$$S_1 = - \sinh^2|r| \ln \frac{\sinh^2|r|}{\cosh^2|r|} + \ln \cosh^2|r|$$

$$= - \sinh^2|r| \ln \sinh^2|r| + \left(\sinh^2|r| + 1 \right) \ln \cosh^2|r|$$

$$= \cosh^2|r| \ln \cosh^2|r| - \sinh^2|r| \ln \sinh^2 r$$

d) $p(n_1) = |\langle n | \rho_1 | n \rangle|^2 = \tanh^{2n}|r| / \cosh^2|r|.$