Exercises section 7

1 a) $[L_i, L_j] = [\epsilon_{ikl}\, r_k p_l,\ \epsilon_{jmn}\, r_m p_n]$

$$= \epsilon_{ikl}\,\epsilon_{jmn}\,[r_k p_l,\ r_m p_n]$$

$$= \epsilon_{ikl}\,\epsilon_{jmn}\left(r_k [p_l, r_m] p_n + r_m [r_k, p_n] p_l\right)$$

$$= \epsilon_{ikl}\,\epsilon_{jmn}\left(-i\hbar\, r_k p_n\, \delta_{lm} + i\hbar\, r_m p_l\, \delta_{kn}\right)$$

$$= -i\hbar\, \epsilon_{ikm}\,\epsilon_{jmn}\, r_k p_n + i\hbar\, \epsilon_{inl}\,\epsilon_{jmn}\, r_m p_l$$

$$= i\hbar\left[\epsilon_{ikm}\,\epsilon_{mjn}\, r_k p_n - \epsilon_{jmn}\,\epsilon_{nil}\, r_m p_l\right]$$

$$= i\hbar\left(\epsilon_{ikm}\,\epsilon_{mjn} - \epsilon_{jkm}\,\epsilon_{min}\right) r_k p_n$$

$$= i\hbar\left(\epsilon_{ijl}\,\epsilon_{lkn}\, r_k p_n\right) = i\hbar\, \epsilon_{ijl} L_l .$$

$[L^2, L_i] = 0$: show for one component :

$$[L^2, L_z] = [L_x^2 + L_y^2 + L_z^2, L_z] = [L_x^2 + L_y^2, L_z] =$$

$$L_x [L_x, L_z] + [L_x, L_z] L_x + L_y [L_y, L_z] + [L_y, L_z] L_y =$$

$$L_x (-i\hbar) L_y + (-i\hbar) L_y L_x + L_y (i\hbar) L_x + i\hbar L_x L_y = 0.$$

$$[L_z, L_\pm] = [L_z, L_x \pm i L_y] = [L_z, L_x] \pm i [L_z, L_y]$$

$$= i\hbar L_y \pm i(-i\hbar) L_x = \pm L_x + i\hbar L_y = \pm L_\pm .$$

b) $L^2|l,l\rangle = l(l+1)\hbar^2|l,l\rangle$ and $[L_-, L^2] = 0$

$L_-L^2|l,l\rangle = l(l+1)\hbar^2 L_-|l,l\rangle \Leftrightarrow$

$L^2 \alpha|l,l-1\rangle = l(l+1)\hbar^2 \alpha|l,l-1\rangle \quad (\alpha = \sqrt{l(l+1)-l(l-1)}\,\hbar) \Leftrightarrow$

$L^2|l,l-1\rangle = l(l+1)\hbar^2|l,l-1\rangle$

Repeat this $l-m-1$ times: $\quad L^2|l,m\rangle = l(l+1)\hbar^2|l,m\rangle.$

2a) $S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

$$[S_x, S_y] = \frac{\hbar^2}{4}\left[\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} - \begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}\right] = 2i\frac{\hbar^2}{4}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = i\hbar S_z$$

$$[S_x, S_z] = \frac{\hbar^2}{4}\left[\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} - \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right] = 2i\frac{\hbar^2}{4}\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix} = -i\hbar S_y$$

$$[S_y, S_z] = \frac{\hbar^2}{4}\left[\begin{pmatrix}0 & +i\\ i & 0\end{pmatrix} - \begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix}\right] = 2i\frac{\hbar^2}{4}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = i\hbar S_x.$$

b) S_z must be diagonal with eigenvalues m_s:

$$S_z = \begin{pmatrix}1 & & \\ & 0 & \\ & & -1\end{pmatrix}$$

In this order, the ladder operators are given by

$$S_+ = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix} \quad \text{and} \quad S_- = \begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$$

$$S_x = S_+ + S_- = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y = \frac{S_+ - S_-}{i} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$[S_x, S_y] = \begin{pmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & i \end{pmatrix} = 2i \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \quad \text{etc.}$$

c) $$\exp[-i\vec{\theta}\cdot\vec{\sigma}] = \exp[-i(\theta_x \sigma_x + \theta_y \sigma_y + \theta_z \sigma_z)]$$

$$= \exp\left[-i \underbrace{\begin{pmatrix} \theta_z & \theta_x - i\theta_y \\ \theta_x + i\theta_y & -\theta_z \end{pmatrix}}_{\text{Hermitian } 2\times 2 \text{ matrix}}\right]$$

Therefore the exponent is a unitary 2×2 matrix.

3a) Same Lie algebra means the same eigenvalues, eigenvectors and operator behaviour.

b) $I = \frac{1}{2}$: nucleons $\quad |p\rangle = |\tfrac{1}{2}, \tfrac{1}{2}\rangle$, $|n\rangle = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$

$I = 1$: pions $\quad |\pi^+\rangle = |1,1\rangle$, $|\pi^0\rangle = |1,0\rangle$, $|\pi^-\rangle = |1,-1\rangle$

$I = \frac{3}{2}$: delta baryons $\quad |\Delta^{++}\rangle = |\tfrac{3}{2}, \tfrac{3}{2}\rangle$, $|\Delta^+\rangle = |\tfrac{3}{2}, \tfrac{1}{2}\rangle$

$|\Delta^0\rangle = |\tfrac{3}{2}, -\tfrac{1}{2}\rangle$, $|\Delta^-\rangle = |\tfrac{3}{2}, -\tfrac{3}{2}\rangle$.

c) $|\Delta^{++}\rangle \to |p\rangle \otimes |\pi^+\rangle$

$|\Delta^+\rangle \to |p\rangle \otimes |\pi^0\rangle \quad$ or $\quad |\Delta^+\rangle \to |n\rangle \otimes |\pi^+\rangle$

$|\Delta^0\rangle \to |p\rangle \otimes |\pi^-\rangle \quad$ or $\quad |\Delta^0\rangle \to |n\rangle \otimes |\pi^0\rangle$

$|\Delta^-\rangle \to |n\rangle \otimes |\pi^-\rangle$

d) $$I_-^{(\Delta)} |\Delta^{++}\rangle = I_-^{(\Delta)} |\tfrac{3}{2}, \tfrac{3}{2}\rangle = \sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1) - \tfrac{3}{2}(\tfrac{3}{2}-1)}\, |\tfrac{3}{2}, \tfrac{1}{2}\rangle = \sqrt{3}\, |\Delta^+\rangle$$

$$(I_-^{(n)} + I_-^{(\pi)}) |p, \pi^+\rangle = I_-^{(n)} |\tfrac{1}{2}, \tfrac{1}{2}; 1, 1\rangle + I_-^{(\pi)} |\tfrac{1}{2}, \tfrac{1}{2}; 1, 1\rangle$$

$$= \sqrt{\tfrac{3}{4} - (-\tfrac{1}{4})}\, |\tfrac{1}{2}, -\tfrac{1}{2}; 1, 1\rangle + \sqrt{2 - 0}\, |\tfrac{1}{2}, \tfrac{1}{2}; 1, 0\rangle = |n, \pi^+\rangle + \sqrt{2}\, |p, \pi^0\rangle$$

$$\sqrt{3}\,|\Delta^+\rangle = |n,\pi^+\rangle + \sqrt{2}\,|p,\pi^0\rangle \quad \text{or}$$

$$|\Delta^+\rangle = \frac{|n,\pi^+\rangle + \sqrt{2}\,|p,\pi^0\rangle}{\sqrt{3}}$$

Relative decay ratio is 1:2.

$$I_-^{(\Delta)}\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{15}{4} - \left(-\tfrac{1}{4}\right)}\,\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle = 2\,|\Delta^0\rangle$$

$$\left(I_-^{(n)} + I_-^{(\pi)}\right)\frac{|n,\pi^+\rangle + \sqrt{2}\,|p,\pi^0\rangle}{\sqrt{3}} =$$

$$= \left(\underbrace{I_-^{(n)}|n,\pi^+\rangle}_{0} + \sqrt{2}\,I_-^{(n)}|p,\pi^0\rangle + I_-^{(\pi)}|n,\pi^+\rangle + \sqrt{2}\,I_-^{(\pi)}|p,\pi^0\rangle\right)/\sqrt{3}$$

$$= \left(\sqrt{2}\,|n,\pi^0\rangle + \sqrt{2}\,|n,\pi^0\rangle + \sqrt{2}\cdot\sqrt{2}\,|p,\pi^-\rangle\right)/\sqrt{3}$$

$$= \frac{2\sqrt{2}\,|n,\pi^0\rangle + 2\,|p,\pi^-\rangle}{\sqrt{3}} = 2\,|\Delta^0\rangle$$

$$\text{Or}\quad |\Delta^0\rangle = \frac{|p,\pi^-\rangle + \sqrt{2}\,|n,\pi^0\rangle}{\sqrt{3}}, \quad \text{decay ratio 1:2.}$$

$$\text{Finally}\quad I_-^{(\Delta)}|\Delta^0\rangle = \sqrt{\tfrac{15}{4} + \tfrac{1}{2}\left(-\tfrac{3}{2}\right)}\,\left|\tfrac{3}{2},-\tfrac{3}{2}\right\rangle = \sqrt{3}\,|\Delta^-\rangle$$

$$\left(I_-^{(n)} + I_-^{(\pi)}\right)\frac{|p,\pi^-\rangle + \sqrt{2}\,|n,\pi^0\rangle}{\sqrt{3}} = \frac{|n,\pi^-\rangle}{\sqrt{3}} + \frac{\sqrt{2}\sqrt{2}\,|n,\pi^-\rangle}{\sqrt{3}} = \sqrt{3}\,|n,\pi$$

Or $|\Delta^-\rangle = |n,\pi^-\rangle$, as it should.

4 a) "Good" quantum numbers means that the eigenstates of L, S, commute with the Hamiltonian.

$$H = g\hbar \vec{L}\cdot\vec{S}$$

$$[\vec{L}_z, H] = g\hbar\,[L_z, L_xS_x + L_yS_y + S_zL_z]$$

$$= g\hbar\,[L_z, L_x]S_x + g\hbar\,[L_z, L_y]S_y$$

$$= g\hbar\,(i\hbar L_yS_x - i\hbar L_xS_y) \neq 0$$

$$[L^2, H] = g\hbar\,[L^2, L_xS_x + L_yS_y + L_zS_z]$$

$$= \sum_j g\hbar\,\underbrace{[L^2, L_j]}_{0}\,S_j = 0$$

So l is good, but m is <u>not</u>.

$\vec{J} = \vec{L} + \vec{S}$, $[\vec{J}^2, H] = 0$ so j is good.

$$[L_z + S_z, H] = [L_z, H] + [S_z, H]$$

$$= ig\hbar^2\,(L_yS_x - L_xS_y + S_yL_x - S_xL_y) = 0$$

So both j and m_j are good quantum numbers.

b) First-order perturbation theory $E^{(1)}_{lms} = \langle\psi_{lms}|\, g\hbar\vec{L}\cdot\vec{S}\,|\psi_{lms}\rangle$

$$|\psi_{lms}\rangle = |l, m\rangle|s, m_s\rangle$$

$$E^{(1)} = \langle l,m;\, s,m_s \,|\, g\hbar\, \vec{L}\cdot\vec{S} \,|\, l,m;\, s,m_s \rangle$$

$$= g\hbar \langle l,m;\, s,m_s | \Big(\tfrac{1}{4}(L_+ + L_-)(S_+ + S_-) + \tfrac{1}{4i}(L_+ - L_-)(S_+ - S_-)$$

$$+ L_z S_z \Big) | l,m;\, s,m_s \rangle$$

$$= \frac{g\hbar}{4} \langle \psi | \Big(L_+S_+ + L_+S_- + L_-S_+ + L_-S_- - iL_+S_+ + iL_+S_- + iL_-S_+$$

$$- iL_-S_- + 4L_zS_z \Big) | \psi \rangle$$

Notice that $\langle l,m | L_\pm | l,m \rangle = 0$, so most terms are zero.

$$E^{(1)} = g\hbar \langle \psi | L_z S_z | \psi \rangle = g\hbar^3 m\, m_s .$$

c) $H_{mm'ss'} = \langle lm;\, sm_s \,|\, g\hbar\, \vec{L}\cdot\vec{S} \,|\, l,m';\, s,m_s' \rangle$

Now we do <u>not</u> have cancellations, and we need to calculate all

$$H_{mm'm_sm_s'} = \frac{g\hbar}{4} \Big[\langle\psi|L_+S_+|\psi\rangle + \langle\psi|L_+S_-|\psi\rangle + \langle\psi|L_-S_+|\psi\rangle$$

$$+ \langle\psi|L_-S_-|\psi\rangle - i\langle\psi|L_+S_+|\psi\rangle + i\langle\psi|L_+S_-|\psi\rangle$$

$$+ i\langle\psi|L_-S_+|\psi\rangle - i\langle\psi|L_-S_-|\psi\rangle + \cancel{m'm_s'}$$

$$+ 4\langle\psi|L_zS_z|\psi\rangle \Big] \qquad (*)$$

$$\langle\psi|L_+S_+|\psi\rangle = \hbar^2 \langle l,m;\, s,m_s | \sqrt{l(l+1) - m(m+1)} \cdot \sqrt{s(s+1) - m_s(m_s+1)}$$

$$\times | l,m';\, s,m_s' \rangle$$

$$= \hbar^2 \sqrt{l(l+1) - m(m+1)} \sqrt{s(s+1) - m_s(m_s+1)}\; \delta_{m,m'-1}\, \delta_{m_s,m_s'-1}$$

$$\langle\psi|L_+S_-|\psi\rangle = \hbar^2\sqrt{l(l+1)-m(m+1)}\sqrt{s(s+1)-m_s(m_s-1)}\;\delta_{m,m'-1}\,\delta_{m_s,m_s'+1}$$

$$\langle\psi|L_-S_+|\psi\rangle = \hbar^2\sqrt{l(l+1)-m(m-1)}\sqrt{s(s+1)-m_s(m_s+1)}\;\delta_{m,m'+1}\,\delta_{m_s,m_s'-1}$$

$$\langle\psi|L_-S_-|\psi\rangle = \hbar^2\sqrt{l(l+1)-m(m-1)}\sqrt{s(s+1)-m_s(m_s-1)}\;\delta_{m,m'+1}\,\delta_{m_s,m_s'+1}$$

$$\langle\psi|L_zS_z|\psi\rangle = \hbar^2 m\, m_s\;\delta_{m,m'}\,\delta_{m_s,m_s'}$$

Substitute in (*) to obtain the matrix elements.

5a) spin $\frac{3}{2}$: 4D Hilbert space

spin 2 : 5D Hilbert space

Total Hilbert space is $4 \times 5 = 20$ D.

From (7.67):

$$5 \otimes 4 = 8 \oplus 6 \oplus 4 \oplus 2$$

So four multiplets with largest = 8D.

b) $n \otimes n = (2n-1) \oplus (2n-3) \oplus (2n-5) \oplus \ldots$

n multiplets. for n = even <u>and</u> odd.