Exercises section 5

1. $\rho = \frac{1}{2}|01\rangle\langle 01| + \frac{1}{2}|10\rangle\langle 10|$

$$= \frac{1}{2}|0\rangle\langle 0| \otimes |1\rangle\langle 1| + \frac{1}{2}|1\rangle\langle 1| \otimes |0\rangle\langle 0| \qquad (*)$$

$$|0\rangle\langle 0| = \frac{1}{2}\left(|+\rangle + |-\rangle\right)\left(\langle +| + \langle -|\right)$$
$$= \frac{1}{2}\left(|+\rangle\langle +| + |+\rangle\langle -| + |-\rangle\langle +| + |-\rangle\langle -|\right)$$

$$|1\rangle\langle 1| = \frac{1}{2}\left(|+\rangle - |-\rangle\right)\left(\langle +| - \langle -|\right)$$
$$= \frac{1}{2}\left(|+\rangle\langle +| - |+\rangle\langle -| - |-\rangle\langle +| + |-\rangle\langle -|\right)$$

Substitute this into $(*)$ to obtain Eq. (5.15).

$\rho_B = \mathrm{Tr}_A(\overset{|+\rangle_A\langle +|}{\rho})$: force off-diagonal terms to zero:

$$\rho_B = \frac{1}{4}\left[|+\rangle\langle +| + |+\rangle\langle +| + |-\rangle\langle -| + |-\rangle\langle -| - 0 - 0 + 0 + 0\right]$$
$$= \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| = \frac{I}{2}.$$

2a) $|\psi_{00}\rangle = |\Phi^+\rangle$, $|\psi_{01}\rangle = |\Phi^-\rangle$, $|\psi_{10}\rangle = |\Psi^+\rangle$, $|\psi_{11}\rangle = |\Psi^-\rangle$.

$$|\psi_{nm}\rangle = \frac{1}{\sqrt{2}}\sum_\ell (-1)^{\ell m}|\ell, \ell+n\rangle = \frac{1}{\sqrt{2}}\sum_\ell (-1)^{\ell m} X_2^n |\ell,\ell\rangle$$
$$= \frac{1}{\sqrt{2}}\sum_\ell X_2^n Z_2^m |\ell,\ell\rangle = X_2^n Z_2^m |\psi_{00}\rangle.$$

Teleportation: ${}_{12}\langle \psi_{jk}|\psi, \psi_{00}\rangle_{123} = Z_3^k X_3^j |\psi\rangle_3$

b) $\langle \psi_{jk} | \psi, \psi_{nm} \rangle = \langle \psi_{jk} | X_3^n Z_3^m | \psi, \psi_{00} \rangle = X_3^n Z_3^m \langle \psi_{jk} | \psi, \psi_{00} \rangle$

$= X_3^n Z_3^m Z_3^k X_3^j | \psi \rangle_3 = (-1)^{j(m+k)} X_3^{n+j} Z_3^{m+k} | \psi \rangle_3 .$

Since an overall minus sign is unobservable:

$${}_{12}\langle \psi_{jk} | \psi, \psi_{nm} \rangle_{123} = X_3^{n+j} Z_3^{m+k} | \psi \rangle_3$$

c) $\langle \psi_{kl} | \psi_{nm} \rangle = \frac{1}{N} \sum_{rj} e^{2\pi i (rn - jk)/N} \langle j, j+l | r, r+m \rangle$

$= \frac{1}{N} \sum_{rj} e^{2\pi i (rn - jk)/N} \delta_{jr} \langle j+l | r+m \rangle$

$= \frac{1}{N} \sum_{r} e^{2\pi i r(n-k)/N} \delta_{lm} = \delta_{nk} \delta_{lm} .$

d) Define $X|j\rangle = |j+1\rangle$ and $Z|j\rangle = e^{2\pi i j/N} |j\rangle .$

$\langle \psi_{jk} | \psi, \psi_{00} \rangle$ with $|\psi\rangle = \sum_l c_l |l\rangle$

$\frac{1}{\sqrt{N}} \langle \psi_{jk} | \sum_{lm} c_l | l, m, m \rangle = \frac{1}{\sqrt{N}} \sum_{lm} c_l \underbrace{\langle \psi_{jk} | l, m \rangle}_{\text{evaluate}} |m\rangle .$

$\langle l, m | \psi_{jk} \rangle = \frac{1}{\sqrt{N}} \sum_r e^{2\pi i r j/N} \langle l, m | r, r+k \rangle$

$= \frac{1}{\sqrt{N}} \sum_r e^{2\pi i r j/N} \delta_{rl} \langle m | r+k \rangle$

$= \frac{1}{\sqrt{N}} e^{2\pi i j l/N} \delta_{m, l+k}$

$$\langle \psi_{jk} | \psi_1 \psi_{00} \rangle = \sum_{lm} e^{-2\pi i j l/N} \delta_{m,l+k} c_l |m\rangle$$

$$= \sum_l e^{-2\pi i j l/N} c_l |l+k\rangle$$

$$= \sum_l e^{-2\pi i j l/N} c_l X^k |l\rangle$$

$$= \sum_l c_l X^k Z^{-j} |l\rangle = X^k Z^{-j} |\psi\rangle$$

$$|\psi_{nm}\rangle = X_2^m Z_2^n |\psi_{00}\rangle$$

$${}_{12}\langle \psi_{jk} | \psi_1 \psi_{nm} \rangle_{123} = X_3^m Z_3^n X_3^k Z_3^{-j} |\psi\rangle.$$

4 a) $\mathrm{Tr}_A\left(|0\rangle\langle 0| \otimes |\Phi^-\rangle\langle\Phi^-|\right) = \frac{1}{2}|1\rangle\langle 1| \rightarrow \rho_B = |1\rangle\langle 1|$

b) $\mathrm{Tr}_A\left(\left[(1-\epsilon)|0\rangle\langle 0| + \epsilon|1\rangle\langle 1|\right]|\Phi^-\rangle\langle\Phi^-|\right) = \frac{1}{2}\left((1-\epsilon)|1\rangle\langle 1| + \epsilon|0\rangle\langle 0|\right)$

$\rho_B = (1-\epsilon)|1\rangle\langle 1| + \epsilon|0\rangle\langle 0|$

c) $\mathrm{Tr}_A\left((E_0+E_1)|\Phi^-\rangle\langle\Phi^-|\right) = \mathrm{Tr}_A\left(|\Phi^-\rangle\langle\Phi^-|\right)$, since we "forget" the measurement outcomes E_0 and E_1.

$E_0 + E_1 = \mathbb{I} \Rightarrow E_1 = \epsilon|0\rangle\langle 0| + (1-\epsilon)|1\rangle\langle 1|$.