Exercises section 4

1. a) $|\psi\rangle = \frac{1}{2}|0\rangle + \frac{1}{2}|+\rangle = \frac{1+\sqrt{2}}{2\sqrt{2}}|0\rangle + \frac{1}{2\sqrt{2}}|1\rangle$

Show that $\||\psi\rangle\| \neq 1$:

$$\langle\psi|\psi\rangle = \left(\frac{1+\sqrt{2}}{2\sqrt{2}}\langle 0| + \frac{1}{2\sqrt{2}}\langle 1|\right)\left(\frac{1+\sqrt{2}}{2\sqrt{2}}|0\rangle + \frac{1}{2\sqrt{2}}|1\rangle\right)$$

$$= \left(\frac{1+\sqrt{2}}{2\sqrt{2}}\right)^2\langle 0|0\rangle + \left(\frac{1}{2\sqrt{2}}\right)^2\langle 1|1\rangle$$

$$= \frac{3}{8} + \frac{2\sqrt{2}}{8} + \frac{1}{8} \neq 1.$$

b) $\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|+\rangle\langle +|$ on the basis $\{|0\rangle, |1\rangle\}$:

$|0\rangle\langle 0| = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ and $|+\rangle\langle +| = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$

$$\rho = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \frac{1}{4}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} = \begin{pmatrix}\frac{3}{4} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{4}\end{pmatrix}$$

$Tr(\rho) = \frac{3}{4} + \frac{1}{4} = 1$ (add diagonals)

Any density operator in a convex sum over (possibly non-orthogonal) one-dimensional projection operators, (P_j) and the probabilities add to one.

Any P_j can always be written as $\begin{pmatrix}1 & & 0\\ & 0 & \\ 0 & & \ddots\end{pmatrix}$ in some orthonormal basis $\Rightarrow Tr(P_j) = 1$.

$\rho = \sum_j p_j P_j$ with $\sum_j p_j = 1$ since these are probabilities.

$Tr(\rho) = Tr\left(\sum_j p_j P_j\right) = \sum_j p_j Tr(P_j) = \sum_j p_j = 1.$

P_j is also Hermitian (again obvious from matrix).

$$\rho^\dagger = \Big(\sum_j p_j P_j\Big)^\dagger = \sum_j p_j^* P_j^\dagger = \sum_j p_j P_j = \rho .$$

c) $\rho = w_1 \rho_1 + w_2 \rho_2$ and $Tr(\rho_i) = 1$, $\rho_i^\dagger = \rho_i$, $\langle \rho_i \rangle \geq 0$. $i = 1, 2$.

$$Tr(\rho) = w_1 Tr(\rho_1) + w_2 Tr(\rho_2) = w_1 + w_2 = 1$$

$$\rho^\dagger = (w_1 \rho_1 + w_2 \rho_2)^\dagger = w_1^* \rho_1^\dagger + w_2^* \rho_2^\dagger = w_1 \rho_1 + w_2 \rho_2 = \rho .$$

$$\langle \psi | \rho | \psi \rangle = w_1 \underbrace{\langle \psi | \rho_1 | \psi \rangle}_{\geq 0} + w_2 \underbrace{\langle \psi | \rho_2 | \psi \rangle}_{\geq 0} \geq 0 .$$

d) $$Tr(\rho A) = \sum_{jk} p_j \langle \phi_k | \psi_j \rangle \langle \psi_j | A | \phi_k \rangle = \sum_j p_j \langle \psi_j | A | \psi_j \rangle$$

$$= \sum_\ell \lambda_\ell \langle \lambda_\ell | A | \lambda_\ell \rangle .$$

All measured quantities are expectation values. Even though ρ was produced as a statistical mixture of states $|\psi_j\rangle$, we cannot distinguish between this and the spectral resolution in <u>any</u> way. The difference is therefore <u>unphysical</u>. As a consequence, in quantum mechanics you cannot ascribe <u>reality</u> to the particular decomposition of the density operator.

2. $\langle\psi|A|\psi\rangle = \int dx\, \langle\psi|x\rangle\langle x|A|\psi\rangle = \int dx\, \psi^*(x)\, A\, \psi(x)$

for pure wavefunctions.

Mixed: $\sum_j p_j \langle\psi_j|A|\psi_j\rangle = \sum_j \int dx\, p_j\, \psi_j^*(x)\, A\, \psi_j(x)$

or, if the distribution is a continuum:

$\langle A\rangle = \int du\, \pi(u)\, \langle\psi(u)|A|\psi(u)\rangle = \int dx\, du\, \pi(u)\, \psi^*(x,u)\, A\psi(x,u)$.

3. $$P^2 = \begin{pmatrix} |a|^2 & a\bar{b} \\ \bar{a}b & |b|^2 \end{pmatrix}\begin{pmatrix} |a|^2 & a\bar{b} \\ \bar{a}b & |b|^2 \end{pmatrix} = \begin{pmatrix} |a|^4+|a|^2|b|^2 & a\bar{b}|a|^2 + a\bar{b}|b|^2 \\ \bar{a}b|a|^2+\bar{a}b|b|^2 & |a|^2|b|^2+|b|^4 \end{pmatrix}$$

$$= \begin{pmatrix} |a|^2(|a|^2+|b|^2) & a\bar{b}(|a|^2+|b|^2) \\ \bar{a}b(|a|^2+|b|^2) & |b|^2(|a|^2+|b|^2) \end{pmatrix} = \begin{pmatrix} |a|^2 & a\bar{b} \\ \bar{a}b & |b|^2 \end{pmatrix} = P.$$

4. Define $a_i = e^{-i\gamma_i t}$ for convenience. The eigenvalues of ρ can then be written as

$$\lambda = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1-2a_1+a_1^2+a_2^2}\,.$$

Since $0 \leq \lambda \leq 1$, we require that $1-2a_1+a_1^2+a_2^2 < 1$:

$a_1^2+a_2^2-2a_1 < 0$.

violation: $\gamma_1 = 2\gamma_2 + 2\epsilon$, which leads to $a_1 = a_2^2(1-\delta)$
or $a_1(1+\delta) = a_2^2$ (to order 1 in δ).

$a_1^2 + a_1(1+\delta) - 2a_1 < 0 \iff a_1(a_1 - 1 + \delta) < 0$

Since $a_1 = e^{-\gamma_1 t} < 1$, we must have $\delta \to 0$.

5 a) $H = \sum_n E_n |E_n\rangle\langle E_n|$

b) $\rho = \dfrac{\sum_n e^{-E_n/kT} |E_n\rangle\langle E_n|}{\sum_n e^{-E_n/kT}}$

$= \dfrac{e^{-H/kT}}{Tr[e^{-H/kT}]}$

c) Directly: $\langle E\rangle = Tr[\rho H] = \dfrac{Tr[e^{-H/kT} H]}{Tr[e^{-H/kT}]}$

$\langle E\rangle = \dfrac{Tr[\sum_n E_n |E_n\rangle\langle E_n| \times \sum_m e^{-E_m/kT} |E_m\rangle\langle E_m|]}{\sum_j e^{-E_j/kT}}$

$= \dfrac{\sum_n E_n e^{-E_n/kT}}{\sum_m e^{-E_m/kT}}$. $(*)$

$Z = Tr(e^{-H/kT})$

$\ln Z = \ln(Tr[e^{-H/kT}]) = \ln(\sum_n e^{-E_n/kT})$

$\dfrac{\partial \ln Z}{\partial (kT)} = \dfrac{1}{\sum_n e^{-E_n/kT}} \dfrac{\partial}{\partial (kT)^{-1}} \left(\sum_n e^{-E_n/kT} \right)$

$= - \dfrac{\sum_n E_n e^{-E_n/kT}}{\sum_m e^{-E_m/kT}}$. agreement with $(*)$

d) Harmonic oscillator: $E_n = n\omega\hbar$: $\sum_n e^{-n\hbar\omega/kT} = \dfrac{1}{1-e^{-\frac{\hbar\omega}{kT}}}$

$\ln Z = -\ln(1 - e^{-\hbar\omega/kT})$. $T \to 0$ means $S \to 0$.

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a) A perfect photodectector tells us exactly how many photons are measured, so the outcomes are integers n.

$$p(n) = \mathrm{Tr}\left[|\psi\rangle\langle\psi| \cdot P_n \right] \quad \text{with} \quad P_n = |n\rangle\langle n|$$

$$p(n) = \langle n|\psi\rangle\langle\psi|n\rangle = |c_n|^2 .$$

b) No photons: only $n=0$ contributes:

$$p(0) = |c_0|^2 .$$

All the rest: $\quad 1 - p(0) = 1 - |c_0|^2$

c) Outcome "no photons" only when all photons are lost

$$p(0) = |c_0|^2 + (1-\eta)|c_1|^2 + (1-\eta)^2 |c_2|^2 + \dots$$
$$= \sum_{n=0}^{\infty} (1-\eta)^n |c_n|^2$$

all the rest $\quad 1 - p(0) = 1 - \sum_{n=0}^{\infty} (1-\eta)^n |c_n|^2 = \sum_{n=0}^{\infty} |c_n|^2 - \sum_{n=0}^{\infty} (1-\eta)^n |c_n|^2$

$$= \sum_{n=0}^{\infty} \left[1 - (1-\eta)^n \right] |c_n|^2 .$$

d) dark counts, afterpulsing, jitter.

a) Possible outcomes: $\uparrow$ and $\downarrow$ with electron in state $|\uparrow\rangle$ and $|\downarrow\rangle$ respectively.

b) Possible outcomes $\uparrow, \downarrow$, and "fail".

$$E_{fail} = p\,\mathbb{I}, \quad E_{\uparrow} = (1-p)\,|\uparrow\rangle\langle\uparrow|, \quad E_{\downarrow} = (1-p)\,|\downarrow\rangle\langle\downarrow|$$

with $E_{\uparrow} + E_{\downarrow} + E_{fail} = \mathbb{I}$.

c) $p(\uparrow) = (1-p)|\alpha|^2, \quad p(\downarrow) = (1-p)|\beta|^2, \quad p(fail) = p.$

$$\rho_{fail} = \frac{p\left(|\uparrow\rangle\langle\uparrow|\rho|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\rho|\downarrow\rangle\langle\downarrow|\right)}{p} = |\alpha|^2|\uparrow\rangle\langle\uparrow| + |\beta|^2|\downarrow\rangle\langle\downarrow|$$

coherence lost.