Exercises section 3

1. $H_H = U^{\dagger}(t)\, H_s\, U(t) = \exp\left[\frac{i}{\hbar} H_s t\right] H_s \exp\left[-\frac{i}{\hbar} H_s t\right] = H_s$.

2 a) $\langle\alpha|\alpha\rangle = e^{-|\alpha|^2} \sum\limits_{n,m=0}^{\infty} \frac{\alpha^{*m}\,\alpha^{n}}{\sqrt{m!\,n!}} \langle m|n\rangle = e^{-|\alpha|^2} \sum\limits_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = 1$.

b) $U(t)|\alpha\rangle = \exp\left[-\frac{i}{\hbar} Ht\right]|\alpha\rangle = e^{-|\alpha|^2/2} \sum\limits_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-\frac{i}{\hbar}Ht} |n\rangle$

Use $e^{-iHt/\hbar}|n\rangle = e^{-\frac{i}{\hbar}\hbar\omega(n+\frac{1}{2})t} |n\rangle$

$$U(t)|\alpha\rangle = e^{-|\alpha|^2/2 - i\omega t/2} \sum_{n=0}^{\infty} \frac{\alpha^n e^{-in\omega t}}{\sqrt{n!}} |n\rangle$$

$$= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}} |n\rangle = |\alpha e^{-i\omega t}\rangle .$$

c) $a^{\dagger}a|n\rangle = a^{\dagger}\sqrt{n}\,|n-1\rangle = \sqrt{n}\sqrt{n-1+1}\,|n-1+1\rangle = n\,|n\rangle$.

d) $|n\rangle = \frac{a^{\dagger}}{\sqrt{n}} |n-1\rangle \Rightarrow |n\rangle = \frac{a^{\dagger n}}{\sqrt{n!}} |0\rangle$

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n a^{\dagger n}}{n!} |0\rangle .$$

e) $U(t)|\alpha\rangle = |\alpha e^{-i\omega t}\rangle$

$$= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n\, U(t)\, a^{\dagger n}}{n!} |0\rangle$$

$$= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} U(t)\, a^{\dagger n}\, U^{\dagger}(t) \underbrace{U(t)\,|0\rangle}_{|0\rangle}$$

$$= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \left(U(t)\, a^{\dagger}\, U^{\dagger}(t)\right)^n |0\rangle$$

$$= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \; a^{\dagger n} \, e^{-in\omega t} \, |0\rangle$$

Therefore $U(t) \, a^\dagger \, U^\dagger(t) = a^\dagger e^{-i\omega t}$.

f) $\hat{x} \propto \dfrac{a + a^\dagger}{\sqrt{2}}$ and $\hat{p} \propto \dfrac{i(a - a^\dagger)}{\sqrt{2}}$.

Substitute $a \to a e^{i\omega t}$ and $a^\dagger \to a^\dagger e^{-i\omega t}$:

$\hat{x}(t) = \cos\omega t \; \hat{x} + \sin\omega t \; \hat{p}$ and.

$\hat{p}(t) = -\sin\omega t \; \hat{x} + \cos\omega t \; \hat{p}$.

3. $S_\pm = X \pm iY$, $\quad S_\pm(t) = e^{\pm i\omega t} S_\pm(0)$.

$$Y(t) = \frac{S_+ - S_-}{2i} = \frac{1}{2i}\left(e^{i\omega t} S_+(0) - e^{-i\omega t} S_-(0)\right)$$

$$= \frac{1}{2i}\left(e^{i\omega t} X(0) + i e^{i\omega t} Y(0) - e^{-i\omega t} X(0) + i e^{-i\omega t} Y(0)\right)$$

$$= \frac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right) X(0) + \frac{1}{2i}\left(i e^{i\omega t} + i e^{-i\omega t}\right) Y(0)$$

$$= \sin\omega t \; X(0) + \cos\omega t \; Y(0)$$

$A = a_0 \mathbb{I} + a_x X + a_y Y + a_z Z \qquad [H, Z] = 0 \Rightarrow Z(t) = Z(0)$

$$A(t) = a_0 \mathbb{I} + a_x X(t) + a_y Y(t) + a_z Z.$$

$$= a_0 \mathbb{I} + a_z Z + (a_x \cos\omega t + a_y \sin\omega t) X(0)$$

$$+ (-a_x \sin\omega t + a_y \cos\omega t) Y(0)$$

$$= \begin{pmatrix} a_0 + a_z & a_x \cos + a_y \sin - i a_y \cos + i a_x \sin \\ a_x \cos + a_y \sin + i a_y \cos - i a_x \sin & a_0 - a_z \end{pmatrix}$$

$$= \begin{pmatrix} a_0 + a_z & (a_x - i a_y) e^{i\omega t} \\ (a_x + i a_y) e^{-i\omega t} & a_0 - a_z \end{pmatrix}$$

$$A^\dagger(t) = \begin{pmatrix} a_0^* + a_z^* & (a_x^* - i a_y^*) e^{i\omega t} \\ (a_x^* + i a_y^*) e^{-i\omega t} & a_0^* - a_z^* \end{pmatrix}$$

If $a_\mu^* = a_\mu$, then $A^\dagger(t) = A(t)$.

4 a) $_S\langle\psi(t)| A_S |\psi(t)\rangle_S = {}_I\langle\psi(t)| \underbrace{U_0^\dagger(t) A_S U_0(t)}_{A_I(t)} |\psi(t)\rangle_I$

$A_I(t) = U_0^\dagger(t) A_S U_0(t)$.

b) $i\hbar \frac{d}{dt} |\psi(t)\rangle_S = H_S |\psi(t)\rangle_S$ (Schrödinger eq.) $\Leftrightarrow$

$i\hbar \frac{d}{dt} U_0(t) |\psi(t)\rangle_I = H_S U_0(t) |\psi(t)\rangle_I \Leftrightarrow$

$i\hbar \left(\frac{dU_0}{dt}\right) |\psi(t)\rangle_I + i\hbar U_0(t) \frac{d}{dt} |\psi(t)\rangle_I = H_S U_0(t) |\psi(t)\rangle_I \Leftrightarrow$

$\frac{i\hbar}{i\hbar} H_0 U_0 |\psi(t)\rangle_I + i\hbar U_0 \frac{d}{dt} |\psi(t)\rangle_I = H_S U_0 |\psi(t)\rangle_I \Leftrightarrow$

$U_0^\dagger H_0 U_0 |\psi(t)\rangle_I + i\hbar \frac{d}{dt} |\psi(t)\rangle_I = U_0^\dagger H_S U_0 |\psi(t)\rangle_I \Leftrightarrow$

$U_0^\dagger H_0 U_0 |\psi(s)\rangle_I + i\hbar \frac{d}{dt} |\psi(t)\rangle_I = \left(H_0 + U_0^\dagger V U_0\right) |\psi(t)\rangle_I \Leftrightarrow$

$i\hbar \frac{d}{dt} |\psi(t)\rangle_I = U_0^\dagger V U_0 |\psi(t)\rangle_I \equiv H_I(t) |\psi(t)\rangle_I$.

So $H_I(t) = U_0^\dagger V U_0 \neq U^\dagger (H_0 + V) U \qquad H_I \neq H_S, H_H$

5a) $H|\psi\rangle = E|\psi\rangle$ and $[H,T] = i\hbar$

$$H e^{i\varpi T}|\psi\rangle = e^{i\varpi T}\underbrace{e^{-i\varpi T} H e^{i\varpi T}}|\psi\rangle$$

$$e^{-i\varpi T} H e^{i\varpi T} = H - i\varpi[T,H] - \frac{(i\varpi)^2}{2!}[T,\underbrace{[T,H]]}_{0.} + \dots$$

$$= H - \hbar\varpi$$

Therefore:

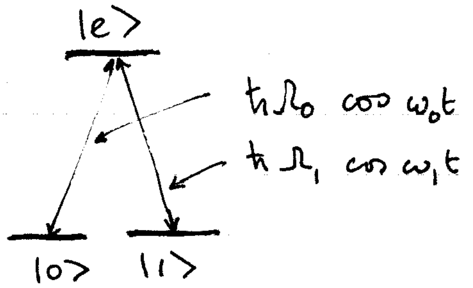
$$H e^{i\varpi T}|\psi\rangle = e^{i\varpi T}(H - \hbar\varpi)|\psi\rangle = (E-\hbar\varpi)\, e^{i\varpi T}|\psi\rangle.$$

b) $e^{i\varpi T}|\psi\rangle$ is an eigenstate of H with eigenvalue $H - \hbar\varpi$. Since $\varpi \in \mathbb{R}$, the eigenvalues of H range from $-\infty$ to $+\infty$.

c) The operator T with $[H,T] = i\hbar$ is unphysical.

6 a) Free Hamiltonian : $H_0 = \sum_i \epsilon_i |\epsilon_i\rangle\langle\epsilon_i|$

$\epsilon_0 = 0$, $\epsilon_1 = 0$, $\epsilon_e = \hbar\omega$: $H_0 = \hbar\omega |e\rangle\langle e|$



The coupling makes the system jump from $|0\rangle$ or $|1\rangle$ to $|e\rangle$, and vice versa

$$H_I = \hbar\cos\omega_0 t\left(\Omega_0 |0\rangle\langle e| + \Omega_0^* |e\rangle\langle 0| + \hbar\cos\omega_1 t\left(\Omega_1 |1\rangle\langle e| + \Omega_1^* |e\rangle\langle 1|\right)\right.$$

$$H = H_0 + H_I = \begin{pmatrix} 0 & 0 & \hbar\Omega_0^*\cos\omega_0 t \\ 0 & 0 & \hbar\Omega_1^*\cos\omega_1 t \\ \hbar\Omega_0\cos\omega_0 t & \hbar\Omega_1\cos\omega_1 t & \hbar\omega \end{pmatrix}.$$

b) $H|\psi\rangle = i\hbar\partial_t|\psi\rangle$ and in rotating frame $H'|\psi'\rangle = i\hbar\partial_t|\psi'\rangle$

with $|\psi\rangle = U^\dagger|\psi'\rangle$

$$H U^\dagger|\psi'\rangle = i\hbar\partial_t\left(U^\dagger|\psi'\rangle\right) = i\hbar\left(\partial_t U^\dagger\right)|\psi'\rangle + i\hbar U^\dagger\partial_t|\psi'\rangle$$

Apply U to the left on both sides:

$$U H U^\dagger|\psi'\rangle = \left[i\hbar\, U(\partial_t U^\dagger) + i\hbar\partial_t\right]|\psi'\rangle$$

$$= \left[i\hbar\, U(\partial_t U^\dagger) + H'\right]|\psi'\rangle$$

So $H' = U H U^\dagger - i\hbar\, U(\partial_t U^\dagger)$, as required.

c) $H' = UHU^\dagger - i\hbar\, U(\partial_t U^\dagger)$

$$U(\partial_t U^\dagger) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i(\omega_0-\omega_1)t} & 0 \\ 0 & 0 & e^{-i\omega_0 t} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & i(\omega_0-\omega_1)e^{+i(\omega_0-\omega_1)t} & 0 \\ 0 & 0 & i\omega_0 e^{+i\omega_0 t} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & i(\omega_0-\omega_1) & 0 \\ 0 & 0 & i\omega_0 \end{pmatrix}$$

$$UHU^\dagger = \begin{pmatrix} 0 & 0 & e^{-i\omega_0 t}\hbar\Omega_0^* \cos\omega_0 t \\ 0 & 0 & e^{-i\omega_1 t}\hbar\Omega_1^* \cos\omega_1 t \\ e^{i\omega_0 t}\hbar\Omega_0 \cos\omega_0 t & e^{i\omega_1 t}\hbar\Omega_1 \cos\omega_1 t & \hbar\omega \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} 0 & 0 & \Omega_0^* + \Omega_0^* e^{-2i\omega_0 t} \\ 0 & 0 & \Omega_1^* + \Omega_1^* e^{-2i\omega_1 t} \\ \Omega_0 + \Omega_0 e^{2i\omega_0 t} & \Omega_1 + \Omega_1 e^{2i\omega_1 t} & 2\omega \end{pmatrix}$$

$$\approx \frac{\hbar}{2}\begin{pmatrix} 0 & 0 & \Omega_0^* \\ 0 & 0 & \Omega_1^* \\ \Omega_0 & \Omega_1 & 2\omega \end{pmatrix}$$ since $e^{i\phi}$ averages to 0 when $t \gg \frac{1}{\omega}$.

$$H' = \begin{pmatrix} 0 & 0 & \hbar\Omega_0^*/2 \\ 0 & \hbar(\omega_0-\omega_1) & \hbar\Omega_1^*/2 \\ \hbar\Omega_0/2 & \hbar\Omega_1/2 & \hbar\omega_0 + \hbar\omega \end{pmatrix}$$

d) Eigenvalues of H': $-\lambda^3 + (\omega_0+\omega)\lambda^2 + (|\Omega_0|^2+|\Omega_1|^2)\lambda = 0$

$$\lambda_3 = 0, \qquad \lambda_\pm = \frac{\omega_0+\omega}{2} \pm \sqrt{\left(\frac{\omega_0+\omega}{2}\right)^2 + |\Omega_0|^2 + |\Omega_1|^2}.$$

$$H'\vec{v}_0 = 0 \cdot \vec{v}_0 = 0$$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 0 & \Omega_0^* \\ 0 & 0 & \Omega_1^* \\ \Omega_0 & \Omega_1 & 2(\omega+\omega_0) \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$\Rightarrow \quad c = 0 \quad \text{and} \quad a\Omega_0 + b\Omega_1 = 0$$

choose $a = \Omega_1$ and $b = -\Omega_0$, then normalize:

$$|\vec{v}_0\rangle = \frac{\Omega_1}{\sqrt{|\Omega_0|^2+|\Omega_1|^2}}|0\rangle - \frac{\Omega_0}{\sqrt{|\Omega_0|^2+|\Omega_1|^2}}|1\rangle$$

Set $\frac{\Omega_1}{\sqrt{|\Omega_0|^2+|\Omega_1|^2}} \equiv \cos\theta$: $\underline{|\vec{v}_0\rangle = \cos\theta\,|0\rangle - \sin\theta\,|1\rangle}$.

e) Start in state $|0\rangle$ with $\Omega_0 = 0$ and Ω_1 some value.

Then slowly increase Ω_0 and decrease Ω_1.

When $\Omega_1 = 0$, the ~~state~~ system will be in state $|1\rangle$.

By <u>not</u> populating state $|e\rangle$, you are not susceptible to spontaneous emission from $|e\rangle$ to the ground state. $\rightarrow$ more control.