Exercises section 2

1. $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\det(X - \lambda \mathbb{I}) = \det \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0$

eigenvalues: $(\lambda+1)(\lambda-1) = 0$ or $\lambda = \pm 1$.

eigenvectors: $X\psi = \lambda\psi$: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \pm\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{pmatrix} a = \pm b \\ b = \pm a \end{pmatrix}$

Normalisation: $|a|^2 + |b|^2 = 1 \Rightarrow |a| = |b| = \frac{1}{\sqrt{2}}$

$\psi_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\psi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$\psi_1 \cdot \psi_2 = \frac{1}{2}(1,1)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2}(1 \cdot 1 - 1 \cdot 1) = 0$

$\psi_1 \cdot \psi = \psi_2 \cdot \psi_2 = 1$. Therefore $\{\psi_1, \psi_2\}$ is an orthonormal basis.

$$\begin{aligned} \langle\psi|X|\psi\rangle &= \left(\alpha^*\langle 0| + \beta^*\langle 1|\right)\left(X\alpha|0\rangle + X\beta|1\rangle\right) \\ &= \left(\alpha^*\langle 0| + \beta^*\langle 1|\right)\left(\alpha|1\rangle + \beta|0\rangle\right) \\ &= \alpha^*\beta + \beta^*\alpha = 2\,\mathrm{Re}(\alpha^*\beta). \end{aligned}$$

2. $A|\psi\rangle = a|\psi\rangle \Rightarrow \langle A\rangle = a$

$A^2|\psi\rangle = aA|\psi\rangle = a^2|\psi\rangle \Rightarrow \langle A^2\rangle = a^2$

$(\Delta A)^2 = \langle\psi|(A - \langle A\rangle)^2|\psi\rangle = \langle A^2\rangle - \langle A\rangle^2 = a^2 - a^2 = 0$.

3. $A = A^\dagger \rightarrow \langle\psi|A|\psi\rangle = \langle\psi|A^\dagger|\psi\rangle^* = \langle\psi|A|\psi\rangle$ for all $|\psi\rangle$.

Let $A|\phi_j\rangle = a_j|\phi_j\rangle$, then $\langle\phi_j|A|\phi_j\rangle = a_j\langle\phi_j|\phi_j\rangle = \langle\phi_j|A|\phi_j\rangle^* = a_j^*$

Therefore $a_j = a_j^*$

Converse: Let $|\psi\rangle = \sum_j c_j|\phi_j\rangle$, then $\langle\psi|A|\psi\rangle = \sum_j a_j|c_j|^2$

$a_j \in \mathbb{R} \rightarrow \langle\psi|A|\psi\rangle$ real for all $|\psi\rangle \rightarrow A = A^\dagger$.

4. $|0,i\rangle \xrightarrow[u]{} |0,0\rangle$ and $|1,i\rangle \xrightarrow[u]{} |1,1\rangle$

Must work for an arbitrary state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

$$|\psi, i\rangle \Rightarrow \alpha|0,i\rangle + \beta|1,i\rangle \xrightarrow[u]{} \alpha|0,0\rangle + \beta|1,1\rangle$$

However, $|\psi,\psi\rangle = (\alpha|0\rangle+\beta|1\rangle)(\alpha|0\rangle+\beta|1\rangle)$

$$= \alpha^2|0,0\rangle + \alpha\beta|0,1\rangle + \alpha\beta|1,0\rangle + \beta^2|1,1\rangle$$

So $\alpha|0,0\rangle + \beta|1,1\rangle \neq |\psi,\psi\rangle$, and $|\psi\rangle$ cannot be copied unless we know α and β.

5a). Write the Schwarz-inequality as

$$\langle f|f\rangle\langle g|g\rangle \geq |\langle f|g\rangle|^2 \geq \frac{1}{4}|\langle f|g\rangle + \langle g|f\rangle|^2$$

$$\langle f|f\rangle = \langle\psi|(A-\langle A\rangle)(A-\langle A\rangle)|\psi\rangle = \langle\psi|(A-\langle A\rangle)^2|\psi\rangle = (\Delta A)^2$$

$$\langle g|g\rangle = (\Delta B)^2$$

$$\langle f|g\rangle = i\langle\psi|(A-\langle A\rangle)(B-\langle B\rangle)|\psi\rangle$$

$$= i\langle\psi|(AB - A\langle B\rangle - B\langle A\rangle + \langle A\rangle\langle B\rangle)|\psi\rangle$$

$$\langle g|f\rangle = -i\langle\psi|(BA - B\langle A\rangle - A\langle B\rangle + \langle A\rangle\langle B\rangle)|\psi\rangle$$

$$\langle f|g\rangle + \langle g|f\rangle = i\langle\psi|[A,B]|\psi\rangle$$

Substitute this back into the inequality gives

$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}|\langle\psi|[A,B]|\psi\rangle|^2 \qquad \square.$$

b) Canonically conjugate observers: $[A,B] = i\hbar$

$(\Delta A)^2(\Delta B)^2 \geq \frac{\hbar^2}{4}$ or $\Delta A\Delta B \geq \frac{\hbar}{2}$.

c) No, because there is no time operator.

6. $\det(H - \lambda I) = \det\begin{vmatrix} -\lambda & iE & 0 \\ -iE & -\lambda & 0 \\ 0 & 0 & -E-\lambda \end{vmatrix} = -(E+\lambda)(\lambda^2 - E^2) = 0$

$\lambda = E$ and $\lambda = -E$ (2x).

$$\langle H \rangle = \frac{E}{5}(1+i,\ 1+i,\ 1)\begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1-i \\ 1-i \\ 1 \end{pmatrix}$$

$$= \frac{E}{5}(1+i,\ 1+i,\ 1)\begin{pmatrix} i+1 \\ -i-1 \\ -1 \end{pmatrix} = \frac{E}{5}\left(2i - 2i - 1\right) = -\frac{E}{5}.$$

7. $H = \frac{p^2}{2m} + V(x)$ and $V(x) = \sum_{n=0}^{\infty} v_n x^n$

$[H, p] = \left[\frac{p^2}{2m} + V(x),\ p\right] = 0$ required

$= [V(x), p] = \sum_{n=0}^{\infty} v_n [x^n, p]$

$[x^n, p] = i n x^{n-1}$

$[H, p] = \sum_{n=0}^{\infty} v_n [x^n, p] = i \sum_{n=0}^{\infty} n v_n x^{n-1}$

Since all x^n are linearly independent, $[H, p]$ is zero if and only if all $v_n = 0$ for $n > 0$.

This is a <u>constant</u> potential. $\rightarrow$ No force on the particle

since $F = -\frac{\partial V}{\partial x}$.

<u>Momentum is conserved</u> when no force acts on a particle

$\downarrow$

$[H, p] = 0.$