Exercises Section 1

1 a) Calculate $x a + y b + z c = 0$: (*)

$$\begin{pmatrix} 2x \\ 3x \\ -x \end{pmatrix} + \begin{pmatrix} 0 \\ y \\ 2y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -5z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Three equations:

$2x = 0 \longrightarrow x = 0$

$3x + y = 0 \longrightarrow 3\cdot 0 + y = 0 \longrightarrow y = 0$

$-x + 2y - 5z = 0 \longrightarrow z = 0.$

So (*) is zero only if $x = y = z = 0$. Therefore, a, b, c are linearly independent.

b) $\langle\psi|\chi\rangle = \left(-3i\langle\phi_1| + 7i\langle\phi_2|\right)\left(|\phi_1\rangle + 2|\phi_2\rangle\right)$

$= -3i\langle\phi_1|\phi_1\rangle + 14i\langle\phi_2|\phi_2\rangle = 11i$

$\langle\psi|\psi\rangle = \left(-3i\langle\phi_1| + 7i\langle\phi_2|\right)\left(3i|\phi_1\rangle - 7i|\phi_2\rangle\right)$

$= 9 + 49 = 58.$

$\langle\chi|\chi\rangle = \left(\langle\phi_1| + 2\langle\phi_2|\right)\left(|\phi_1\rangle + |\phi_2\rangle\right) = 1 + 4 = 5.$

$|\langle\psi|\chi\rangle|^2 \leq \langle\psi|\psi\rangle\langle\chi|\chi\rangle$

$|11i|^2 \leq 58 \cdot 5$ or $121 \leq 290$.

c) First we calculate A^{-1} via the augmented matrix method (easiest):

$$\left(\begin{array}{ccc|ccc} 0 & i & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & i & 2 & 1 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & -i & 0 \end{array}\right) \rightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{i}{2} & 0 \end{array}\right) \qquad B = \begin{pmatrix} 2 & i & 0 \\ 3 & 1 & 5 \\ 0 & -i & -2 \end{pmatrix}$$

$$\uparrow$$
$$A^{-1}$$

$$A^{-1}B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ \frac{1}{2} & -\frac{i}{2} & 0 \end{pmatrix} \begin{pmatrix} 2 & i & 0 \\ 3 & 1 & 5 \\ 0 & -1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -2 \\ 3 & 1 & 5 \\ 1-\frac{3i}{2} & 0 & -\frac{5i}{2} \end{pmatrix}$$

not the same.

$$BA^{-1} = \begin{pmatrix} 2 & i & 0 \\ 3 & 1 & 5 \\ 0 & -1 & -2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ \frac{1}{2} & -\frac{i}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & i & 2 \\ \frac{5}{2} & 1-\frac{5i}{2} & 3 \\ -1 & -1+i & 0 \end{pmatrix}$$

So $A^{-1}B \neq BA^{-1}$.

d) $$A \otimes B = \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$B \otimes A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

2a) Hermitian: $A + A^\dagger$, $i(A - A^\dagger)$, and $A^\dagger A$.
Note the role of 'i' in the adjoint.

b) A shared eigenbasis means:

$$A = \sum_j a_j |\phi_j\rangle\langle\phi_j| \quad \text{and} \quad B = \sum_j b_j |\phi_j\rangle\langle\phi_j| \qquad \langle\phi_j|\phi_k\rangle = \delta_{jk}.$$

$$AB = \sum_{jk} a_j b_k |\phi_j\rangle\langle\phi_j|\phi_k\rangle\langle\phi_k| = \sum_j a_j b_j |\phi_j\rangle\langle\phi_j|$$

$$BA = \sum_{jk} b_j a_k |\phi_j\rangle\langle\phi_j|\phi_k\rangle\langle\phi_k| = \sum_j b_j a_j |\phi_j\rangle\langle\phi_j|$$

$$AB - BA = \sum_j \underbrace{(a_j b_j - b_j a_j)}_{0 \text{ (numbers)}} |\phi_j\rangle\langle\phi_j| = 0 = [A, B].$$

c) $U|\phi_n\rangle = |\psi_n\rangle$ and $\langle\phi_n|\phi_m\rangle = \langle\psi_n|\psi_m\rangle = \delta_{nm}$ (given)

unitary $:= U^\dagger = U^{-1}$

$U^\dagger U = \mathbb{I}$: $\langle\psi_m|\psi_n\rangle = \left(\langle\phi_m|U^\dagger\right)\left(U|\phi_n\rangle\right) = \langle\phi_m|\underbrace{U^\dagger U}_{\mathbb{I}}|\phi_n\rangle = \delta_{mn}$

$$UU^\dagger = \mathbb{I} : \langle\phi_m|\phi_n\rangle = \sum_j \langle\phi_m|\psi_j\rangle\langle\psi_j|\phi_n\rangle$$

$$= \sum_j \langle\phi_m|U|\phi_j\rangle\langle\phi_j|U^\dagger|\phi_n\rangle$$

$$= \langle\phi_m|UU^\dagger|\phi_n\rangle = \delta_{mn} \Rightarrow UU^\dagger = \mathbb{I}.$$ QED.

d) First, look at 2×2 matrices:

$\begin{pmatrix} o & o \\ o & o \end{pmatrix}$ real: 2 parameters

complex (but conjugates of each other): 2 parameters.

Determinant 1: $2+2-1 = 3$ real parameters.

$d\times d$ unitary matrices

$\frac{1}{2}d(d-1)$ complex
d real -1 determinant

$$\left.\begin{matrix}\frac{1}{2}d(d-1) \text{ complex} \\ d \text{ real } -1 \text{ determinant}\end{matrix}\right\} 2\cdot\frac{d}{2}(d-1) + d - 1 = \underline{\underline{d^2-1}}$$

3a) $\det A = -i\left(i\cdot 0 - 2\cdot 1\right) = 2i$, $\operatorname{Tr} A = 0+1+0 = 1$

$\det B = 2\left(1\cdot(-2) - 5\cdot(-i)\right) - 3\left(i(-2) + i\cdot 0\right) = -4+5i+6i = -4+11i$

$\operatorname{Tr} B = 2+1-2 = 1.$

b) $\operatorname{Tr}\left[|\psi\rangle\langle\psi| A\right] = \sum_j \langle\phi_j|\psi\rangle\langle\psi|A|\phi_j\rangle = \sum_j \langle\psi|A|\phi_j\rangle\langle\phi_j|\psi\rangle$

$= \langle\psi|A|\psi\rangle \equiv \langle A\rangle.$

c) $\operatorname{Tr}(A) = \sum_j \langle\phi_j|A|\phi_j\rangle = \sum_{jk} \langle\phi_j|\psi_k\rangle\langle\psi_k|A|\phi_j\rangle$

$= \sum_{jk} \langle\psi_k|A|\phi_j\rangle\langle\phi_j|\psi_k\rangle = \sum_k \langle\psi_k|A|\psi_k\rangle.$

4a) $F(t) = e^{At}e^{Bt} \Rightarrow \frac{dF}{dt} = Ae^{At}e^{Bt} + e^{At}Be^{Bt}$

$\frac{dF}{dt} = Ae^{At}e^{Bt} + Be^{At}e^{Bt} + [e^{At}, B]e^{Bt}$

$= \left(A + e^{At}Be^{-At}\right)F$

b) $G(t) = e^{At + Bt + f(t)H(A,B)}$

$\frac{dG}{dt} = \left(A + B + f'(t)H(A,B)\right)G.$

Use that A, B commute with $[A,B]$ and evaluate Eq. (1.64) for F

$$e^{At} B e^{-At} = B + t[A,B]$$

$$\frac{dF}{dt} = (A + B + t\,[A,B])\,F$$

To find what expression for H and f yields $F = G$, we equate the time derivatives:

$\frac{dF}{dt} = \frac{dG}{dt}$. Set $H = [A,B]$ and $f'(t) = t$

When $t = 0$ $F(t) = G(t) = \mathbb{I} \longrightarrow f(t) = \frac{1}{2}t^2$

Therefore $e^{At} e^{Bt} = e^{At + Bt + \frac{1}{2}t^2[A,B]}$.

Set $t = 1$ to find $e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$.

c) $[A,B]^\dagger = (AB - BA)^\dagger = (AB)^\dagger - (BA)^\dagger = BA - AB = -[A,B]$.

d) Write out:

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] =$$

$$[A,BC] - [A,CB] + [B,CA] - [B,AC] + [C,AB] - [C,BA] =$$

$$ABC - BCA - ACB + CBA + BCA - CAB - BAC + ACB + CAB - ABC - CBA + BAC$$

$= 0$